

\mathcal{N}_p -TYPE FUNCTIONS WITH HADAMARD GAPS IN THE UNIT BALL

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ABSTRACT. We study the holomorphic functions with Hadamard gaps in \mathcal{N}_p -spaces on the unit ball of \mathbb{C}^n when $0 < p \leq n$ and $p > n$. A corollary on analytic functions with Hadamard gaps on \mathcal{N}_p -spaces on the unit disk is also given.

1. INTRODUCTION

Let \mathbb{B} be the open unit ball in \mathbb{C}^n with \mathbb{S} as its boundary and $H(\mathbb{B})$ the collection of all holomorphic functions in \mathbb{B} . H^∞ denotes the Banach space consisting of all bounded holomorphic functions in \mathbb{B} with the norm $\|f\|_\infty = \sup_{z \in \mathbb{B}} |f(z)|$. The Bergman-type space $A^{-p}(\mathbb{B})$ is the space of all $f \in H(\mathbb{B})$ such that

$$\|f\|_p = \sup_{z \in \mathbb{B}} |f(z)|(1 - |z|^2)^p < \infty.$$

Let $A_0^{-p}(\mathbb{B})$ denote the closed subspace of $A^{-p}(\mathbb{B})$ such that $\lim_{|z| \rightarrow 1} |f(z)|(1 - |z|^2)^p = 0$.

The \mathcal{N}_p -space in the unit disk \mathbb{D} was first introduced in [11] and studied in [16], which is defined as, for $p > 0$,

$$\mathcal{N}_p(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_p = \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} |f(z)|^2 (1 - |\sigma_a(z)|^2)^p dA(z) \right)^{1/2} < \infty \right\},$$

where dA is the normalized area measure over \mathbb{D} and $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is a Möbius transformation of \mathbb{D} .

Let dV denote the normalized volume measure over \mathbb{B} and $\Phi_a(z)$ the automorphism of \mathbb{B} for $a \in \mathbb{B}$, i.e.,

$$\Phi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle},$$

where $s_a = \sqrt{1 - |a|^2}$, P_a is the orthogonal projection into the space spanned by a and $Q_a = I - P_a$ (see, e.g., [20]). The \mathcal{N}_p -space on \mathbb{B} was introduced in [5], i.e.,

$$\begin{aligned} \mathcal{N}_p &= \mathcal{N}_p(\mathbb{B}) \\ &= \left\{ f \in H(\mathbb{B}) : \|f\|_p = \sup_{a \in \mathbb{B}} \left(\int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \right)^{1/2} < \infty \right\}. \end{aligned}$$

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The little space of \mathcal{N}_p -space, denoted by \mathcal{N}_p^0 , which consisting of all $f \in \mathcal{N}_p$ such that

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) = 0.$$

In [5], several basic properties of $\mathcal{N}_p(\mathbb{B})$ -spaces are proved, in connection with the Bergman-type spaces A^{-q} . In particular, an embedding theorem for $\mathcal{N}_p(\mathbb{B})$ -spaces and $A^{-q}(\mathbb{B})$ is obtained, together with other useful properties.

Theorem 1.1. [5] *The following statements hold:*

- (a) For $p > q > 0$, we have $H^\infty \hookrightarrow \mathcal{N}_q \hookrightarrow \mathcal{N}_p \hookrightarrow A^{-\frac{n+1}{2}}$.
- (b) For $p > 0$, if $p > 2k - 1$, $k \in (0, \frac{n+1}{2}]$, then $A^{-k} \hookrightarrow \mathcal{N}_p$. In particular, when $p > n$, $\mathcal{N}_p = A^{-\frac{n+1}{2}}$.
- (c) \mathcal{N}_p is a functional Banach space with the norm $\|\cdot\|_p$, and moreover, its norm topology is stronger than the compact-open topology.

An $f \in H(\mathbb{B})$ written in the form

$$f(z) = \sum_{k=0}^{\infty} P_{n_k}(z),$$

where P_{n_k} is a homogeneous polynomial of degree n_k , is said to have *Hadamard gaps* if for some $c > 1$ (see. e.g., [14]),

$$\frac{n_{k+1}}{n_k} \geq c, \quad \forall k \geq 0.$$

Hadamard gaps series on spaces of holomorphic functions in \mathbb{D} or in \mathbb{B} has been studied quite well. We refer the readers to the related results in [1, 2, 3, 4, 7, 9, 10, 14, 15, 17, 18, 19, 21] and the reference therein.

The aim of the present paper is to characterize the holomorphic functions with Hadamard gaps in \mathcal{N}_p -space for two different cases $0 < p \leq n$ and $p > n$. Our main results are contained in Section 2.

Throughout this paper, for $a, b \in \mathbb{R}$, $a \lesssim b$ ($a \gtrsim b$, respectively) means there exists a positive number C , which is independent of a and b , such that $a \leq Cb$ ($a \geq Cb$, respectively). Moreover, if both $a \lesssim b$ and $a \gtrsim b$ hold, then we say $a \simeq b$.

2. MAIN RESULTS AND PROOFS

To formulate our main result, we denote

$$M_k = \sup_{\xi \in \mathbb{S}} |P_{n_k}(\xi)| \quad \text{and} \quad L_k = \left(\int_{\xi \in \mathbb{S}} |P_{n_k}(\xi)|^2 d\sigma(\xi) \right)^{1/2},$$

where $d\sigma$ is the normalized surface measure on \mathbb{S} , that is, $\sigma(\mathbb{S}) = 1$. Clearly for each $k \geq 0$, M_k and L_k are well-defined.

2.1. The case when $0 < p \leq n$. In this subsection, we study the Hadamard gaps series in \mathcal{N}_p -spaces when $0 < p \leq n$. We have the following result.

Theorem 2.1. *Let $0 < p \leq n$ and $f(z) = \sum_{k=0}^{\infty} P_{n_k}(z)$ with Hadamard gaps. Considering the following statements.*

- (a) $\sum_{k=0}^{\infty} \left(\frac{1}{2^{k(1+p)}} \sum_{2^k \leq n_j < 2^{k+1}} M_j^2 \right) < \infty;$
- (b) $f \in \mathcal{N}_p^0;$
- (c) $f \in \mathcal{N}_p;$
- (d) $\sum_{k=0}^{\infty} \left(\frac{1}{2^{k(1+p)}} \sum_{2^k \leq n_j < 2^{k+1}} L_j^2 \right) < \infty.$

We have $(a) \implies (b) \implies (c) \implies (d).$

Proof. • $(a) \implies (b).$ Suppose that (a) holds. First, we prove that $f \in \mathcal{N}_p$. For $f(z) = \sum_{k=0}^{\infty} P_{n_k}(z)$, by using the polar coordinates and [20, Lemma 1.8], we have

$$\begin{aligned}
 \|f\|_p^2 &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left| \sum_{k=0}^{\infty} P_{n_k}(z) \right|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \\
 &\leq \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left(\sum_{k=0}^{\infty} |P_{n_k}(z)| \right)^2 \frac{(1 - |a|^2)^p (1 - |z|^2)^p}{|1 - \langle z, a \rangle|^{2p}} dV(z) \\
 &= \sup_{a \in \mathbb{B}} \left\{ (1 - |a|^2)^p \int_{\mathbb{B}} \left(\sum_{k=0}^{\infty} |P_{n_k}(z)| \right)^2 \frac{(1 - |z|^2)^p}{|1 - \langle z, a \rangle|^{2p}} dV(z) \right\} \\
 &\leq 2n \sup_{a \in \mathbb{B}} \left\{ (1 - |a|^2)^p \int_0^1 \left(\sum_{k=0}^{\infty} |P_{n_k}(r\xi)| \right)^2 (1 - r^2)^p \left(\int_{\mathbb{S}} \frac{1}{|1 - \langle r\xi, a \rangle|^{2p}} d\sigma(\xi) \right) dr \right\} \\
 &= 2n \sup_{a \in \mathbb{B}} \left\{ (1 - |a|^2)^p \int_0^1 \left(\sum_{k=0}^{\infty} |P_{n_k}(\xi)| r^{n_k} \right)^2 (1 - r^2)^p \left(\int_{\mathbb{S}} \frac{1}{|1 - \langle r\xi, a \rangle|^{2p}} d\sigma(\xi) \right) dr \right\} \\
 &\leq 2n \sup_{a \in \mathbb{B}} \left\{ (1 - |a|^2)^p \int_0^1 \left(\sum_{k=0}^{\infty} M_k r^{n_k} \right)^2 (1 - r^2)^p \left(\int_{\mathbb{S}} \frac{1}{|1 - \langle r\xi, a \rangle|^{2p}} d\sigma(\xi) \right) dr \right\}.
 \end{aligned}$$

Applying [20, Theorem 1.12], for each $a \in \mathbb{B}$ and $r \in [0, 1]$, we have

$$\begin{aligned}
 \int_{\mathbb{S}} \frac{1}{|1 - \langle r\xi, a \rangle|^{2p}} d\sigma(\xi) &= \int_{\mathbb{S}} \frac{1}{|1 - \langle \xi, ar \rangle|^{2p}} d\sigma(\xi) \\
 &= \int_{\mathbb{S}} \frac{d\sigma(\xi)}{|1 - \langle ar, \xi \rangle|^{n+(2p-n)}} \\
 &\simeq \begin{cases} \text{bounded in } \mathbb{B}, & 0 < p < \frac{n}{2}, \\ \log \frac{1}{1-r^2|a|^2} \leq \log \frac{1}{1-|a|^2}, & p = \frac{n}{2}, \\ (1 - r^2|a|^2)^{n-2p} \leq (1 - |a|^2)^{n-2p}, & \frac{n}{2} < p < n. \end{cases}
 \end{aligned}$$

It is clear that, for all cases of p , we can have

$$(1 - |a|^2)^p \int_{\mathbb{S}} \frac{1}{|1 - \langle r\xi, a \rangle|^{2p}} d\sigma(\xi) \leq M, \quad a \in \mathbb{B},$$

where M is a positive number independent of both a and r . Now, applying [8, Theorem 1], we have

$$\begin{aligned}\|f\|_p^2 &\leq 2nM \int_0^1 \left(\sum_{k=0}^{\infty} M_k r^{n_k} \right)^2 (1-r^2)^p dr \\ &\simeq 2nM \sum_{k=0}^{\infty} \frac{1}{2^{k(1+p)}} \left(\sum_{2^k \leq n_j < 2^{k+1}} M_j \right)^2.\end{aligned}$$

Since f is in the Hadamard gaps class, there exists a constant $c > 1$ such that $n_{j+1} \geq cn_j$ for all $j \geq 0$. Hence, the maximum number of n_j 's between 2^k and 2^{k+1} is less or equal to $[\log_c 2] + 1$ for $k = 0, 1, 2, \dots$.

Since for every $k \geq 0$, by Cauchy-Schwarz inequality,

$$\left(\sum_{2^k \leq n_j < 2^{k+1}} M_j \right)^2 \leq ([\log_c 2] + 1) \left(\sum_{2^k \leq n_j < 2^{k+1}} M_j^2 \right).$$

Thus, we have

$$(2.1) \quad \|f\|_p^2 \lesssim \sum_{k=0}^{\infty} \left(\frac{1}{2^{k(1+p)}} \sum_{2^k \leq n_j < 2^{k+1}} M_j^2 \right) < \infty,$$

which implies $f \in \mathcal{N}_p$.

Next, we prove that $f \in \mathcal{N}_p^0$. Let

$$A = \int_{\mathbb{B}} (1 - |\Phi_a(z)|^2)^p dV(z).$$

We claim that $A \rightarrow 0$ as $|a| \rightarrow 1^-$. By [20, Lemma 1.2], we have

$$\begin{aligned}\int_{\mathbb{B}} (1 - |\Phi_a(z)|^2)^p dV(z) &= \int_{\mathbb{B}} \frac{(1 - |a|^2)^p (1 - |z|^2)^p}{|1 - \langle z, a \rangle|^{2p}} dV(z) \\ &= (1 - |a|^2)^p \int_{\mathbb{B}} \frac{(1 - |z|^2)^p}{|1 - \langle z, a \rangle|^{n+1+p+p-n-1}} dV(z).\end{aligned}$$

Applying [20, Theorem 1.12], we know

$$\int_{\mathbb{B}} \frac{(1 - |z|^2)^p}{|1 - \langle a, z \rangle|^{n+1+p+p-n-1}} dV(z) \simeq \begin{cases} \text{bounded in } \mathbb{B}, & 0 < p < n+1, \\ \log \frac{1}{1-|a|^2}, & p = n+1, \\ (1 - |a|^2)^{1+n-p}, & p > n+1. \end{cases}$$

It is clear no matter for what case, $A \rightarrow 0$ as $a \rightarrow 1^-$.

Put $f_m(z) = \sum_{k=0}^m P_{n_k}(z)$, $m \in \mathbb{N}$ and $K_m = \max\{M_0, M_1, \dots, M_m\}$. Note that for each $a \in \mathbb{B}$,

$$\begin{aligned}&\int_{\mathbb{B}} |f_m(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \\ &\leq \int_{\mathbb{B}} \left(\sum_{k=0}^m |P_{n_k}(z)| \right)^2 (1 - |\Phi_a(z)|^2)^p dV(z) \\ &\leq m^2 K_m^2 \int_{\mathbb{B}} (1 - |\Phi_a(z)|^2)^p dV(z),\end{aligned}$$

which tends to 0 as $|a| \rightarrow 1^-$. Hence, $f_m \in \mathcal{N}_p^0$. Moreover, by [6, Corollary 2.6], \mathcal{N}_p^0 is closed and the set of all polynomials is dense in \mathcal{N}_p^0 , and hence it suffices to show that $\|f_m - f\|_p \rightarrow 0$ as $m \rightarrow \infty$. By (2.1), we have

$$(2.2) \quad \|f_m - f\|_p^2 \lesssim \sum_{k=m'}^{\infty} \left(\frac{1}{2^{k(1+p)}} \sum_{2^k \leq n_j < 2^{k+1}} M_j^2 \right),$$

where $m' = \left\lceil \frac{m+1}{\lfloor \log_c 2 \rfloor + 1} \right\rceil$. The result follows from condition (a) and (2.2).

- (b) \implies (c). It is obvious.
- (c) \implies (d). Suppose $f \in \mathcal{N}_p$. As the proof in [14, Theorem 1], we have

$$\begin{aligned} \|f\|_p^2 &= \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \left| \sum_{k=0}^{\infty} P_{n_k}(z) \right|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \\ &\geq \int_{\mathbb{B}} \left| \sum_{k=0}^{\infty} P_{n_k}(z) \right|^2 (1 - |z|^2)^p dV(z) \\ &\simeq \int_{\mathbb{S}} \left(\sum_{k=0}^{\infty} \frac{1}{2^{k(1+p)}} \sum_{2^k \leq n_j < 2^{k+1}} |P_{n_k}(\xi)|^2 \right) d\sigma(\xi) \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{2^{k(1+p)}} \sum_{2^k \leq n_j < 2^{k+1}} L_j^2 \right), \end{aligned}$$

which implies the desired result. \square

Remark 2.2. Generally, when $n > 1$, the above conditions in Theorem 2.1 are not equivalent. For example, (d) \nRightarrow (a). Indeed, put

$$f(z) = \sum_{k=0}^{\infty} 2^{\frac{k(p+1)}{2}} z_1^{2^k}, \quad z = (z_1, z_2, \dots, z_n) \in \mathbb{B}.$$

Since $M_k = 2^{\frac{k(p+1)}{2}}$, we have

$$\sum_{k=0}^{\infty} \left(\frac{1}{2^{k(1+p)}} \sum_{2^k \leq n_j < 2^{k+1}} M_k^2 \right) = \infty.$$

On the other hand, by [20, Lemma 1.11], for each $k \geq 0$, we have

$$L_k^2 = 2^{k(p+1)} \int_{z \in \mathbb{S}} |z_1^{2^k}|^2 d\sigma(z) = 2^{k(p+1)} \cdot \frac{(n-1)!(2^k)!}{(n-1+2^k)!} \lesssim \frac{2^{k(p+1)}}{2^{k(n-1)}},$$

which implies

$$\sum_{k=0}^{\infty} \left(\frac{1}{2^{k(1+p)}} \sum_{2^k \leq n_j < 2^{k+1}} L_k^2 \right) \lesssim \sum_{k=0}^{\infty} \frac{1}{2^{k(n-1)}} < \infty.$$

Next, we consider some special cases when all the conditions in Theorem 2.1 are equivalent.

In [13], the authors constructed a sequence of homogeneous polynomial $\{T_k\}_{k \in \mathbb{N}}$ satisfying $\deg(T_k) = k$,

$$(2.3) \quad \sup_{\xi \in \mathbb{S}} |T_k(\xi)| = 1 \quad \text{and} \quad \int_{\xi \in \mathbb{S}} |T_k(\xi)|^2 d\sigma(\xi) \geq \frac{\pi}{2^{2n}}.$$

An immediate corollary of Theorem 2.1 is stated as follows.

Corollary 2.3. *Let $0 < p \leq n$ and $f(z) = \sum_{k=0}^{\infty} a_k T_{n_k}(z)$ with Hadamard gaps, where $a_k \in \mathbb{C}, k \geq 0$. Then the following statements are equivalent.*

- (a) $\sum_{k=0}^{\infty} \left(\frac{1}{2^{k(1+p)}} \sum_{2^k \leq n_j < 2^{k+1}} |a_j|^2 \right) < \infty;$
- (b) $f \in \mathcal{N}_p^0;$
- (c) $f \in \mathcal{N}_p.$

Proof. The desired result follows from the fact that for each $k \geq 0$, $M_k \simeq L_k$. \square

Moreover, letting $n = 1$, we have the following corollary describing the functions in $\mathcal{N}_p(\mathbb{D})$ with Hadamard gaps.

Corollary 2.4. *Let $0 < p \leq 1$ and $f(z) = \sum_{k=0}^{\infty} b_k z^{n_k}$ with Hadamard gaps, where $b_k \in \mathbb{C}, k \geq 0$. Then the following conditions are equivalent.*

- (a) $f \in \mathcal{N}_p(\mathbb{D});$
- (b) $f \in \mathcal{N}_p^0(\mathbb{D});$
- (c) $\sum_{k=0}^{\infty} \left(\frac{1}{2^{k(1+p)}} \sum_{2^k \leq n_j < 2^{k+1}} |b_j|^2 \right) < \infty.$

Note that the result in [11, Theorem 3.3 (a)] is contained in Corollary 2.4.

Proof. The desired result follows from the fact that when $n = 1$, $M_j = L_j = |b_j|$. \square

2.2. The case when $p > n$. By Theorem 1.1, when $p > n$, all \mathcal{N}_p -spaces coincide with $A^{-\frac{n+1}{2}}$. In this subsection, we consider a more general question about the Hadamard gaps series in A^{-l} for any $l > 0$. We have the following result.

Theorem 2.5. *Let $l > 0$ and $f(z) = \sum_{k=0}^{\infty} P_{n_k}(z)$ with Hadamard gaps, where P_{n_k} is a homogeneous polynomial of degree n_k . Then the following assertions hold.*

- (a) $f \in A^{-l}$ if and only if $\sup_{k \geq 1} \frac{M_k}{n_k^l} < \infty;$
- (b) $f \in A_0^{-l}$ if and only if $\lim_{k \rightarrow \infty} \frac{M_k}{n_k^l} = 0.$

Note that the result in [11, Theorem 3.3 (b)] is a particular case of the assertion (a) in Theorem 2.5.

Proof. (a) **Necessity.** Suppose $f \in A^{-l}$. Fix a $\xi \in \mathbb{S}$ and denote

$$f_{\xi}(w) = \sum_{k=0}^{\infty} P_{n_k}(\xi) w^{n_k} = \sum_{k=0}^{\infty} P_{n_k}(\xi w), \quad w \in \mathbb{D}.$$

Since $f \in H(\mathbb{B})$, it known that for a fixed $\xi \in \mathbb{S}$, $f_\xi(w)$ is holomorphic in \mathbb{D} (see, e.g., [12]). Hence, for any $r \in (0, 1)$, we have

$$\begin{aligned}
 (2.4) \quad M_k &= \sup_{\xi \in \mathbb{S}} |P_{n_k}(\xi)| = \sup_{\xi \in \mathbb{S}} \left| \frac{1}{2\pi i} \int_{|w|=r} \frac{f_\xi(w)}{w^{n_k+1}} dw \right| \\
 &= \frac{1}{2\pi} \sup_{\xi \in \mathbb{S}} \left| \int_{|w|=r} \frac{f(\xi w)}{w^{n_k+1}} dw \right| \\
 &\leq \frac{1}{2\pi} \sup_{\xi \in \mathbb{S}} \int_{|w|=r} \frac{|f(\xi w)|}{r^{n_k+1}} |dw| \\
 &\leq \frac{1}{2\pi} \sup_{\xi \in \mathbb{S}} \int_{|w|=r} \frac{|f(\xi w)|(1-|\xi w|^2)^l}{r^{n_k+1}(1-r)^l} |dw| \\
 &\leq \frac{|f|_l}{r^{n_k}(1-r)^l}.
 \end{aligned}$$

In (2.4), letting $r = 1 - \frac{1}{n_k}$, we have

$$M_k \leq \frac{|f|_l \cdot n_k^l}{(1 - \frac{1}{n_k})^{n_k}}.$$

Thus, for each $k \geq 2$,

$$\frac{M_k}{n_k^l} \leq \frac{|f|_l}{(1 - \frac{1}{n_k})^{n_k}} \leq 4|f|_l,$$

which implies that

$$\sup_{k \geq 1} \frac{M_k}{n_k^l} \leq \max \left\{ \frac{M_1}{n_1^l}, 4|f|_l \right\} < \infty.$$

Sufficiency. Suppose that $\sup_{k \geq 1} \frac{M_k}{n_k^l} < \infty$. Then

$$|f(z)| = \left| \sum_{k=0}^{\infty} P_{n_k} \left(\frac{z}{|z|} \right) |z|^{n_k} \right| \leq \sum_{k=0}^{\infty} M_k |z|^{n_k} \lesssim \sum_{k=0}^{\infty} n_k^l |z|^{n_k}.$$

Thus,

$$\frac{|f(z)|}{1-|z|} \lesssim \left(\sum_{k=0}^{\infty} n_k^l |z|^{n_k} \right) \left(\sum_{s=0}^{\infty} |z|^s \right) = \sum_{t=0}^{\infty} \left(\sum_{n_j \leq t} n_j^l \right) |z|^t.$$

Since

$$\lim_{k \rightarrow \infty} \frac{k^l k!}{l(l+1) \dots (l+k)} = \Gamma(l), \quad l > 0,$$

we have

$$\sup_{k \in \mathbb{N}} \left(\frac{k^l k!}{(k+l)(k+l-1) \dots (l+1)} \right) \leq M,$$

where M is a positive number depending on l . Hence, we have for each $k \geq 0$,

$$\begin{aligned}
 (2.5) \quad \frac{k^l}{(-1)^k \binom{-l-1}{k}} &= \frac{k^l k!}{(-1)^k (-l-1)(-l-2) \dots (-l-k)} \\
 &= \frac{k^l k!}{(k+l)(k+l-1) \dots (l+1)} \leq M,
 \end{aligned}$$

where $\binom{\alpha}{k} = \frac{\alpha(\alpha-1) \dots (\alpha-k+1)}{k!}$, $\alpha \in \mathbb{R}$.

Moreover, since f is in Hadamard gaps class, there exists a constant $c > 1$ such that $n_{j+1} \geq cn_j$ for all $j \geq 0$. Hence

$$(2.6) \quad \frac{1}{k^l} \left(\sum_{n_j \leq k} n_j^l \right) \leq \sum_{m=0}^{\infty} \left(\frac{1}{c^l} \right)^m = \frac{c^l}{c^l - 1}.$$

Combining (2.5) and (2.6), we have

$$\frac{k^l}{(-1)^k \binom{-l-1}{k}} \cdot \frac{1}{k^l} \left(\sum_{n_j \leq k} n_j^l \right) \leq \frac{Mc^l}{c^l - 1},$$

which implies

$$(2.7) \quad \sum_{n_j \leq k} n_j^l \leq (-1)^k \binom{-l-1}{k} \frac{Mc^l}{c^l - 1}.$$

Hence, for any $z \in \mathbb{B}$, by (2.7) we have

$$\frac{|f(z)|}{1-|z|} \lesssim \frac{Mc^l}{c^l - 1} \cdot \sum_{t=0}^{\infty} (-1)^t \binom{-l-1}{t} |z|^t = \frac{Mc^l}{c^l - 1} \cdot \frac{1}{(1-|z|)^{l+1}},$$

which implies

$$|f(z)|(1-|z|^2)^l \lesssim \frac{Mc^l}{c^l - 1},$$

and hence $f \in A^{-l}$.

(b) **Necessity.** Suppose $f \in A_0^{-l}$, that is, for any $\varepsilon > 0$, there exists a $\delta \in (0, 1)$, when $\delta < |z| < 1$,

$$|f(z)|(1-|z|^2)^l < \varepsilon.$$

Take $N_0 \in \mathbb{N}$ satisfying $\delta < 1 - \frac{1}{n_k} < 1$ when $k > N_0$. Then for any $k > N_0$ and $r = 1 - \frac{1}{n_k}$, applying the proof in part (a), we have

$$M_k \leq \frac{n_k^l}{(1 - \frac{1}{n_k})^{n_k}} \cdot \sup_{\delta < |z| < 1} |f(z)|(1-|z|^2)^l < \frac{\varepsilon n_k^l}{(1 - \frac{1}{n_k})^{n_k}},$$

which implies

$$\frac{M_k}{n_k^l} \leq \frac{\varepsilon}{(1 - \frac{1}{n_k})^{n_k}} \leq 4\varepsilon, \quad k > N_0.$$

Hence we have $\lim_{k \rightarrow \infty} \frac{M_k}{n_k^l} = 0$.

Sufficiency. Since $\lim_{k \rightarrow \infty} \frac{M_k}{n_k^l} = 0$, it is clear that $\sup_{k \geq 1} \frac{M_k}{n_k^l} < \infty$ and hence by part (a), we have $f \in A^{-l}$. For any $\varepsilon > 0$, there exists a $N_0 \in \mathbb{N}$ satisfying

$$\frac{M_m}{n_m^l} < \varepsilon,$$

when $m > N_0$. For each $m \in \mathbb{N}$, put $f_m(z) = \sum_{k=0}^m P_{n_k}(z)$. Note that

$$\begin{aligned} |f_m(z)|(1-|z|^2)^l &\leq \left(\sum_{k=0}^m |P_{n_k}(z)| \right) (1-|z|^2)^l \\ &= \left(\sum_{k=0}^m \left| P_{n_k} \left(\frac{z}{|z|} \right) |z|^{n_k} \right| \right) (1-|z|^2)^l \\ &\leq K(1-|z|^2)^l \sum_{k=0}^m |z|^{n_k} \leq Km(1-|z|^2)^l, \end{aligned}$$

where $K = \max\{M_0, M_1, M_2, \dots, M_m\}$. Hence, $\lim_{|z| \rightarrow 1^-} |f_m(z)|(1-|z|^2)^l = 0$, that is, for each $m \in \mathbb{N}$, $f_m \in A_0^{-l}$ and hence it suffices to show that $|f_m - f|_l \rightarrow 0$ as $m \rightarrow \infty$. Indeed, for $m > N_0$, we have

$$|f_m(z) - f(z)| = \left| \sum_{k=m+1}^{\infty} P_{n_k}(z) \right| \leq \sum_{k=m+1}^{\infty} M_k |z|^{n_k} \leq \varepsilon \sum_{k=m+1}^{\infty} n_k^l |z|^{n_k}.$$

Applying the proof in part (a), we have

$$\begin{aligned} \frac{|f_m(z) - f(z)|}{1-|z|} &\leq \varepsilon \left(\sum_{k=m+1}^{\infty} n_k^l |z|^{n_k} \right) \left(\sum_{s=0}^{\infty} |z|^s \right) = \varepsilon \sum_{l=n_{m+1}}^{\infty} \left(\sum_{n_{m+1} \leq n_j \leq l} n_j^l \right) |z|^l \\ &\leq \varepsilon \sum_{t=0}^{\infty} \left(\sum_{n_j \leq t} n_j^l \right) |z|^t \leq M' \frac{\varepsilon}{(1-|z|)^{l+1}}, \end{aligned}$$

where M' is a positive number independent of m . Hence, when $m > N_0$, we have $|f_m - f|_l \leq M'\varepsilon$, which implies that $f \in A_0^{-l}$. \square

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